Critical curves of plane Poiseuille flow with slip boundary conditions

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Abstract

We investigate the linear stability of plane Poiseuille flow in 2D under slip boundary conditions. The slip s is defined by the tangential velocity at the wall in units of the maximal flow velocity. As it turns out, the critical Reynolds number depends smoothly on s but increases quite rapidly.

1 Introduction

No-slip boundary conditions are a convenient idealization of the behavior of viscous fluids near walls. In real systems there is always a certain amount of slip which, however, is hard to detect experimentally because of the required space resolution. In high precision measurements Elrick and Emrich [1] detected slip of the order 0.1% in laminar pipe flow with Reynolds numbers of 16 to 4300. The measuring error in [1] was nearly as low as the fluctuations due to Brownian motion. Very recently, Archer et al. [2] observed the existence of slip in plane laminar Couette flow with added polymers.

We examine how the linear instability of the steady plane Poiseuille flow depends on the slip s defined by

$$s := \frac{u_{\text{wall}}}{u_{\max}} \tag{1}$$

where u_{wall} is the tangential velocity at the wall and $u_{\max}$ is the midstream velocity. As boundary conditions we adopt

$$\frac{\partial u}{\partial z} \pm bu = 0, \quad w = 0, \quad \text{at } z = \pm 1 \tag{2}$$

where z is measured in units of the channel half-width. The slip s is implicitly determined by the parameter $b > 0$ with $s \to 0$ in the limit $b \to \infty$.

2 Orr-Sommerfeld equation with slip boundary conditions

The continuity equation in two dimensions is most conveniently satisfied by introducing a stream function $\Psi(x, z, t)$ where x denotes the streamwise direction and z the direction normal to the boundaries (see Fig. 1). The velocity (u, w) is connected to Ψ through

$$u = \frac{\partial \Psi}{\partial z}, \quad w = -\frac{\partial \Psi}{\partial x}. \tag{3}$$

In terms of Ψ the Navier-Stokes equations for plane Poiseuille flow in two dimensions read in dimensionless form

$$\frac{\partial}{\partial t}\Delta\Psi + \frac{\partial \Psi}{\partial z}\frac{\partial \Delta\Psi}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \Delta\Psi}{\partial z} = \frac{1}{R}\Delta^2\Psi. \tag{4}$$

As usually, Ψ is decomposed in the stream function Ψ_b of the steady profile and a Fourier ansatz in x-direction for the disturbance field with the wave number α:

$$\Psi(x, z, t) = \Psi_b(z) + \sum_{q=-\infty}^{\infty} e^{iq\alpha x}\Psi_q(z, t). \tag{5}$$

However, with slip the basic flow is now given by

$$\Psi_b = z - \frac{bs}{6}z^3; \quad s = \frac{2}{2+b}. \tag{6}$$

The linearized part of (4) leads to the Orr-Sommerfeld equation

$$L\Psi_q = R\frac{\partial}{\partial t}(D^2 - q^2\alpha^2)\Psi_q \tag{7}$$

where

$$L = (D^2 - q^2\alpha^2)^2 - i\alpha qR[U(z)(D^2 - q^2\alpha^2) - U''(z)] \tag{8}$$

with $U(z) = \partial\Psi_b/\partial z$ and $D := \frac{\partial}{\partial z}$.

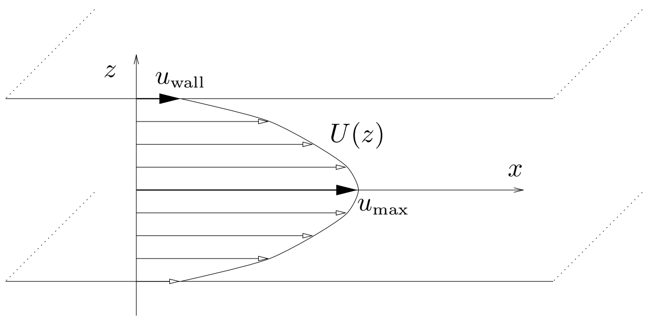

Figure 1: Geometry of the basic flow with slip boundary conditions.

3 Numerical method

We determine the critical (neutral) curves in the parameter space of the Reynolds number R and the wave number α of the disturbance. The Reynolds number is based on the channel half-width and on the midstream velocity of the steady flow.

The solution of the differential equation (7) leads to a generalized eigenvalue problem that we solve numerically as in [3] using up to 70 Chebyshev polynomials as basis functions. The critical curve is the set of points (R, α) for which the most critical eigenvalue has zero real part with all other modes decaying exponentially.

4 Results

In Fig. 2 we present the critical curves for different slips s. The critical Reynolds number R_c is the lowest Reynolds number on the critical curve. We define also the slip s_c by the corresponding normalized tangential velocity of the critical mode at the wall. The results are listed in Tab. 1.

Obviously, the critical Reynolds number depends continuously on s. However, there is, perhaps surprisingly, a strong increase both of R_c and s_c with increasing slip s. In the limit $b \to \infty$, i.e. $s \to 0$, one gets the well-known value $R_c \approx 5772$.

s	0%	0.1%	0.2%	0.5%	1%
s_c	0%	0.9%	1.8%	4.5%	8%
R_c	5772	5773	5781	5847	6070
s	2%	3%	4%	5%	6%
s_c	21%	31%	39%	47%	55%
R_c	6960	8600	11060	15310	23230

Table 1: Slip s_c of the critical mode and critical Reynolds number R_c at different slips s of the steady flow.

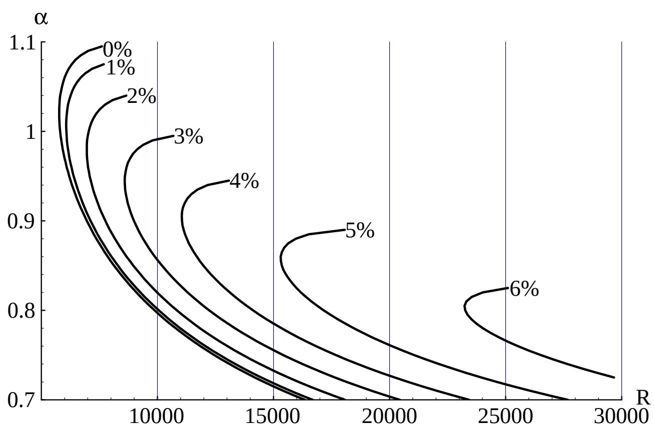

Figure 2: Critical curves of plane Poiseuille flow for different slips $s = 0\%, 1\%, 2\%, 3\%, 4\%, 5\%$ and 6%.

5 Literature

[1] Elrick R.M., Emrich R.J., Phys. Fluids **9** (1966), 28

[2] Archer L.A., Larson R.G., Chen Y.-L., J.Fluid Mech. **301** (1995), 133

[3] Rauh A., Zachrau T., Zoller J., Physica D **86** (1995), 603